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Representations of the vertex operator algebra $V_{L_2}^{A_4}$

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Abstract

The rationality and C_2-cofiniteness of the orbifold vertex operator algebra $V_{L_2}^{A_4}$ are established and all the irreducible modules are constructed and classified. This is part of classification of rational vertex operator algebras with $c = 1$.

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1 Introduction

Motivated by the classification of rational vertex operator algebras with $c = 1$, we investigate the vertex operator algebra $V_{L_2}^{A_4}$ where L_2 is the root lattice of type A_1 and A_4 is the alternating group which is a subgroup of the automorphism group of lattice vertex operator algebra V_{L_2}. The C_2-cofiniteness and rationality of $V_{L_2}^{A_4}$ are obtained, and the irreducible modules are constructed and classified.

Classification of rational vertex operator algebras with $c = 1$ goes back to [G] and [K] in the literature of physics at character level under the assumption that each irreducible character is a modular function over a congruence subgroup and the sum of the square norm of irreducible characters is invariant under the modular group. According to [K], the character of a rational vertex operator algebra with $c = 1$ must be the character of one of the following vertex operator algebras: (a) lattice vertex operator algebras V_L associated with positive definite even lattices L of rank one, (b) orbifold vertex operator algebras V_L^+ under the automorphism of V_L induced from the -1 isometry of L, (c) $V_{\mathbb{Z}\alpha}^G$ where $(\alpha, \alpha) = 2$ and G is a finite subgroup of $SO(3)$ isomorphic to one of $\{A_4, S_4, A_5\}$. As it is pointed out in [DJ1] that this list is not correct if the effective central charge $\tilde{c}$ [DM2] is not equal to c. The vertex operator algebra V_L for any positive definite even lattice L has been characterized by using c, the effective central charge $\tilde{c}$ and the rank of the weigh one subspace as a Lie algebra [DM2]. The orbifold vertex operator algebras V_L^+ for rank one lattices L have also been characterized in [DJ1]-[DJ3] and [ZD]. But the vertex operator algebra $V_{\mathbb{Z}\alpha}^G$ has not been understood well as G is not a cyclic group. Although $V_{\mathbb{Z}\alpha}^G$ is in the above list of rational vertex operator algebras, the rationality of $V_{\mathbb{Z}\alpha}^G$ was unknown. The present paper deals with the case $G = A_4$.

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The main idea is to realize $V_{\mathbb{Z}\alpha}^G$ as $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$ where $(\beta,\beta)=8$ and σ is an automorphism of $sl(2,\mathbb{C})$ of order 3. The vertex operator algebra $V_{\mathbb{Z}\beta}^+$ is well understood (see [DN1]-[DN3], [A1]-[A2]). Also it is easier to deal with the cyclic group $\langle\sigma\rangle$ than nonabelian group A_4. One key step is to give an explicit expression of the generator $u^{(9)}$ of weight 9. Another key step is to prove the C_2-cofiniteness of $V_{\mathbb{Z}\alpha}^{A_4}$. We achieve this by using the fusion rules of the Virasoro vertex operator algebra $L(1,0)$ and technical calculations. The rationality follows from the C_2-cofiniteness [M2]. For the classification of irreducible modules, we follow the standard procedure. We first construct the irreducible σ^i-twisted $V_{\mathbb{Z}\beta}^+$-modules, and then give the irreducible $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$-submodules. According to [M1], these irreducible modules should give a complete list of irreducible $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$-modules.

It is expected that the ideas and techniques developed in this paper will work for $V_{L_2}^{S_4}$ as well. The case $G=A_5$ might be more complicated. Once the rationality of $V_{L_2}^G$ is established for all G, the classification of rational vertex operator algebras with $c=1$ is equivalent to the following conjecture: If V is a simple, rational vertex operator algebra of CFT type such that $\dim V_4<3$ then V is isomorphic to $V_{L_2}^G$ for $G=A_4,S_4,A_5$.

The paper is organized as follows. We recall various notions of twisted modules from [DLM1] in Section 2. We also briefly discuss lattice vertex operator algebras V_L [FLM] and V_L^+ including the classification of irreducible modules and rationality [DN1]-[DN3], [A2], [AD], [DJL]. In Section 3, we identify the vertex operator algebra $V_{L_2}^{A_4}$ with $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$ and discuss several special vectors (which play important roles in later sections) in both $V_{\mathbb{Z}\beta}^+$ and $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$. The rationality and C_2-cofiniteness of $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$ are established in Section 4. The classification of the irreducible $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$-modules is achieved in Section 5.

2 Preliminaries

We first recall weak twisted-modules and twisted-modules for vertex operator algebras from [DLM2]. Let $(V,Y,\mathbf{1},\omega)$ be a vertex operator algebra [B], [FLM] and g an automorphism of V of finite order T. Denote the decomposition of V into eigenspaces with respect to the action of g as

$$V=\oplus_{r\in\mathbb{Z}/T\mathbb{Z}}V^r \tag{2.1}$$

where $V^r=\{v\in V|gv=e^{-2\pi ir/T}v\}$.

Definition 2.1. A *weak g-twisted V-module M* is a vector space equipped with a linear map

$$\begin{aligned}&V\to(\mathrm{End}\,M)\{z\}\\&v\mapsto Y_M(v,z)=\sum_{n\in\mathbb{Q}}v_nz^{-n-1}\quad(v_n\in\mathrm{End}\,M)\end{aligned}$$

which satisfies the following for all $0 \le r \le T-1$, $u \in V^r$, $v \in V$, $w \in M$,

$$Y_M(u,z) = \sum_{n \in \frac{r}{T}+\mathbb{Z}} u_n z^{-n-1} \tag{2.2}$$

$$u_l w = 0 \quad \text{for} \quad l >> 0 \tag{2.3}$$

$$Y_M(\mathbf{1}, z) = 1; \tag{2.4}$$

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(u,z_1)Y_M(v,z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(v,z_2)Y_M(u,z_1) \\ &\quad = z_1^{-1}\left(\frac{z_2+z_0}{z_1}\right)^{r/T} \delta\left(\frac{z_2+z_0}{z_1}\right) Y_M(Y(u,z_0)v, z_2). \end{aligned} \tag{2.5}$$

It is known that (see [DLM2], etc) the twisted-Jacobi identity is equivalent to the following two identities.

$$[u_{m+\frac{r}{T}}, v_{n+\frac{s}{T}}] = \sum_{i=0}^{\infty} \binom{m+\frac{r}{T}}{i} (u_i v)_{m+n+\frac{r+s}{T}-i},$$

$$\sum_{i \ge 0} \binom{s}{i} (u_{m+i}v)_{n+\frac{s+t}{T}-i} = \sum_{i \ge 0} (-1)^i \binom{m}{i} (u_{m+\frac{s}{T}-i} v_{n+\frac{t}{T}+i} - (-1)^m v_{m+n+\frac{t}{T}-i} u_{\frac{s}{T}+i}),$$

where $p, n \in \mathbb{Z}$, $u \in V^s, v \in V^t$.

Definition 2.2. An *admissible g-twisted V-module* $M = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M(n)$ is a $\frac{1}{T}\mathbb{Z}_+$-graded weak g-twisted module such that $u_m M(n) \subset M(\mathrm{wt} u - m - 1 + n)$ for $u \in V$ and $m, n \in \frac{1}{T}\mathbb{Z}$.

Definition 2.3. A *g-twisted V-module* $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$ is a $\mathbb{C}$-graded weak g-twisted V-module with $M_\lambda = \{u \in M | L(0)u = \lambda u\}$ such that M_λ is finite dimensional and for fixed $\lambda \in \mathbb{C}$, $M_{\lambda+n/T} = 0$ for sufficiently small integer n.

We now review the vertex operator algebras $M(1)^+$, V_L^+ and related results from [A1], [A2], [AD], [ADL], [DN1], [DN2], [DN3], [DJL], [FLM].

Let $L = \mathbb{Z}\alpha$ be a positive definite even lattice of rank one. That is, $(\alpha, \alpha) = 2k$ for some positive integer k. Set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend $(\cdot\,,\cdot)$ to a $\mathbb{C}$-bilinear form on $\mathfrak{h}$. Let $\hat{\mathfrak{h}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h} \oplus \mathbb{C}K$ be the affine Lie algebra associated to the abelian Lie algebra $\mathfrak{h}$ so that

$$[\alpha(m),\, \alpha(n)] = 2km\delta_{m+n,0}K \text{ and } [K, \hat{\mathfrak{h}}] = 0$$

for any $m, n \in \mathbb{Z}$, where $\alpha(m) = \alpha \otimes t^m$. Then $\hat{\mathfrak{h}}^{\ge 0} = \mathbb{C}[t] \otimes \mathfrak{h} \oplus \mathbb{C}K$ is a commutative subalgebra. For any $\lambda \in \mathfrak{h}$, we define a one-dimensional $\hat{\mathfrak{h}}^{\ge 0}$-module $\mathbb{C}e^\lambda$ such that $\alpha(m) \cdot e^\lambda = (\lambda, \alpha)\delta_{m,0}e^\lambda$ and $K \cdot e^\lambda = e^\lambda$ for $m \ge 0$. We denote by

$$M(1,\lambda) = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}^{\ge 0})} \mathbb{C}e^\lambda \cong S(t^{-1}\mathbb{C}[t^{-1}]) \text{ (linearly)}$$

the $\hat{\mathfrak{h}}$-module induced from $\hat{\mathfrak{h}}^{\geq 0}$-module $\mathbb{C}e^{\lambda}$. Set

$$M(1) = M(1,0).$$

Then there exists a linear map $Y : M(1) \to \mathrm{End} M(1)[[z, z^{-1}]]$ such that $(M(1), Y, \mathbf{1}, \omega)$ carries a simple vertex operator algebra structure and $M(1,\lambda)$ becomes an irreducible $M(1)$-module for any $\lambda \in \mathfrak{h}$ (see [FLM]). The vacuum vector and the Virasoro element are given by $\mathbf{1} = e^0$ and $\omega = \frac{1}{4k}\alpha(-1)^2\mathbf{1}$, respectively.

Let $\mathbb{C}[L]$ be the group algebra of L with a basis e^{β} for $\beta \in L$. The lattice vertex operator algebra associated to L is given by

$$V_L = M(1) \otimes \mathbb{C}[L].$$

The dual lattice L° of L is

$$L^{\circ} = \{\, \lambda \in \mathfrak{h} \mid (\alpha, \lambda) \in \mathbb{Z} \,\} = \frac{1}{2k} L.$$

Then $L^{\circ} = \cup_{i=-k+1}^{k} (L + \lambda_i)$ is the coset decomposition with $\lambda_i = \frac{i}{2k}\alpha$. In particular, $\lambda_0 = 0$. Set $\mathbb{C}[L + \lambda_i] = \bigoplus_{\beta \in L} \mathbb{C}e^{\beta + \lambda_i}$. Then each $\mathbb{C}[L + \lambda_i]$ is an L-submodule in an obvious way. Set $V_{L+\lambda_i} = M(1) \otimes \mathbb{C}[L + \lambda_i]$. Then V_L is a rational vertex operator algebra and $V_{L+\lambda_i}$ for $i = -k+1, \cdots, k$ are the irreducible modules for V_L (see [B], [FLM], [D1]).

Define a linear isomorphism $\theta : V_{L+\lambda_i} \to V_{L-\lambda_i}$ for $i \in \{-k+1, \cdots, k\}$ by

$$\theta(\alpha(-n_1)\alpha(-n_2)\cdots\alpha(-n_s) \otimes e^{\beta+\lambda_i}) = (-1)^k \alpha(-n_1)\alpha(-n_2)\cdots\alpha(-n_s) \otimes e^{-\beta-\lambda_i}$$

where $n_j > 0$ and $\beta \in L$. Then θ defines a linear isomorphism from $V_{L^{\circ}} = M(1) \otimes \mathbb{C}[L^{\circ}]$ to itself such that

$$\theta(Y(u,z)v) = Y(\theta u, z)\theta v$$

for $u \in V_L$ and $v \in V_{L^{\circ}}$. In particular, θ is an automorphism of V_L which induces an automorphism of $M(1)$.

For any θ-stable subspace U of $V_{L^{\circ}}$, let $U^{\pm}$ be the ± 1-eigenspace of U for θ. Then V_L^{+} is a simple vertex operator algebra.

Also recall the θ-twisted Heisenberg algebra $\mathfrak{h}[-1]$ and its irreducible module $M(1)(\theta)$ from [FLM]. Let χ_s be a character of $L/2L$ such that $\chi_s(\alpha) = (-1)^s$ for $s = 0, 1$ and $T_{\chi_s} = \mathbb{C}$ the irreducible $L/2L$-module with character χ_s. It is well known that $V_L^{T_{\chi_s}} = M(1)(\theta) \otimes T_{\chi_s}$ is an irreducible θ-twisted V_L-module (see [FLM], [D2]). We define actions of θ on $M(1)(\theta)$ and $V_L^{T_{\chi_s}}$ by

$$\theta(\alpha(-n_1)\alpha(-n_2)\cdots\alpha(-n_p)) = (-1)^p \alpha(-n_1)\alpha(-n_2)\cdots\alpha(-n_p)$$

$$\theta(\alpha(-n_1)\alpha(-n_2)\cdots\alpha(-n_p) \otimes t) = (-1)^p \alpha(-n_1)\alpha(-n_2)\cdots\alpha(-n_p) \otimes t$$

for $n_j \in \frac{1}{2} + \mathbb{Z}_+$ and $t \in T_{\chi_s}$. We denote the ± 1-eigenspaces of $M(1)(\theta)$ and $V_L^{T_{\chi_s}}$ under θ by $M(1)(\theta)^{\pm}$ and $(V_L^{T_{\chi_s}})^{\pm}$ respectively. We have the following results:

Theorem 2.4. *Any irreducible module for the vertex operator algebra $M(1)^+$ is isomorphic to one of the following modules:*

$$M(1)^+, M(1)^-, M(1,\lambda) \cong M(1,-\lambda)\ (0 \neq \lambda \in \mathfrak{h}), M(1)(\theta)^+, M(1)(\theta)^-.$$

Theorem 2.5. *Any irreducible V_L^+-module is isomorphic to one of the following modules:*

$$V_L^{\pm}, V_{\lambda_i+L}(i \neq k), V_{\lambda_k+L}^{\pm}, (V_L^{T_{\chi_s}})^{\pm}.$$

Theorem 2.6. *V_L^+ is rational.*

We remark that the classification of irreducible modules for arbitrary $M(1)^+$ and V_L^+ are obtained in [DN1]-[DN3] and [AD]. The rationality of V_L^+ is established in [A2] for rank one lattice and [DJL] in general.

We next turn our attention to the fusion rules of vertex operator algebras. Let V be a vertex operator algebra, and W^i $(i = 1,2,3)$ be ordinary V-modules. We denote by $I_V\begin{pmatrix} W^3 \\ W^1\, W^2 \end{pmatrix}$ the vector space of all intertwining operators of type $\begin{pmatrix} W^3 \\ W^1\, W^2 \end{pmatrix}$. For a V-module W, let W' denote the graded dual of W. Then W' is also a V-module [FHL]. It is well known that fusion rules have the following symmetry (see [FHL]).

Proposition 2.7. *Let W^i $(i = 1,2,3)$ be V-modules. Then*

$$\dim I_V\begin{pmatrix} W^3 \\ W^1\, W^2 \end{pmatrix} = \dim I_V\begin{pmatrix} W^3 \\ W^2\, W^1 \end{pmatrix}, \quad \dim I_V\begin{pmatrix} W^3 \\ W^1\, W^2 \end{pmatrix} = \dim I_V\begin{pmatrix} (W^2)' \\ W^1\, (W^3)' \end{pmatrix}.$$

Recall that $L(c,h)$ is the irreducible highest weight module for the Virasoro algebra with central charge c and highest weight h for $c,h \in \mathbb{C}$. It is well known that $L(c,0)$ is a vertex operator algebra. The following two results were obtained in [M] and [DJ1].

Theorem 2.8. *(1) We have*

$$\dim I_{L(1,0)}\begin{pmatrix} L(1,k^2) \\ L(1,m^2)\, L(1,n^2) \end{pmatrix} = 1, \quad k \in \mathbb{Z}_+,\ |n-m| \leq k \leq n+m,$$

$$\dim I_{L(1,0)}\begin{pmatrix} L(1,k^2) \\ L(1,m^2)\, L(1,n^2) \end{pmatrix} = 0, \quad k \in \mathbb{Z}_+,\ k < |n-m| \text{ or } k > n+m,$$

where $n, m \in \mathbb{Z}_+$.

(2) For $n \in \mathbb{Z}_+$ such that $n \neq p^2$, for all $p \in \mathbb{Z}_+$, we have

$$\dim I_{L(1,0)}\begin{pmatrix} L(1,n) \\ L(1,m^2)\, L(1,n) \end{pmatrix} = 1,$$

$$\dim I_{L(1,0)}\begin{pmatrix} L(1,k) \\ L(1,m^2)\, L(1,n) \end{pmatrix} = 0,$$

for $k \in \mathbb{Z}_+$ such that $k \neq n$.

3 The vertex operator subalgebra $V_{L_2}^{A_4}$

Let $L_2 = \mathbb{Z}\alpha$ be the rank one positive-definite even lattice such that $(\alpha, \alpha) = 2$ and V_{L_2} the associated simple rational vertex operator algebra. Then $(V_{L_2})_1 \cong sl_2(\mathbb{C})$ and $(V_{L_2})_1$ has an orthonormal basis:

$$x^1 = \frac{1}{\sqrt{2}}\alpha(-1)\mathbf{1},\ x^2 = \frac{1}{\sqrt{2}}(e^{\alpha} + e^{-\alpha}),\ x^3 = \frac{i}{\sqrt{2}}(e^{\alpha} - e^{-\alpha}).$$

Let $\tau_i \in Aut(V_{L_2})$, $i = 1, 2, 3$ be such that

$$\tau_1(x^1, x^2, x^3) = (x^1, x^2, x^3)\begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix},$$

$$\tau_2(x^1, x^2, x^3) = (x^1, x^2, x^3)\begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix},$$

$$\tau_3(x^1, x^2, x^3) = (x^1, x^2, x^3)\begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}.$$

Let $\sigma \in Aut(V_{L_2})$ be such that

$$\sigma(x^1, x^2, x^3) = (x^1, x^2, x^3)\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}.$$

Then σ and $\tau_i, i = 1, 2, 3$ generate a finite subgroup of $Aut(V_{L_2})$ isomorphic to the alternating group A_4. We simply denote this subgroup by A_4. It is easy to check that the subgroup K generated by τ_i, $i = 1, 2, 3$ is a normal subgroup of A_4 of order 4. Let

$$J = h(-1)^4\mathbf{1} - 2h(-3)h(-1)\mathbf{1} + \frac{3}{2}h(-2)^2\mathbf{1}, \quad E = e^{\beta} + e^{-\beta}$$

where $h = \frac{1}{\sqrt{2}}\alpha$, $\beta = 2\alpha$. The following lemma comes from [DG].

Lemma 3.1. *$V_{L_2}^K \cong V_{\mathbb{Z}\beta}^+$, and $V_{\mathbb{Z}\beta}^+$ is generated by J and E. Moreover, $(V_{L_2}^K)_4$ is four dimensional with a basis $L(-2)^2\mathbf{1}, L(-4)\mathbf{1}, J, E$.*

By Lemma 3.1, we have $V_{L_2}^{A_4} = (V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$. A direct calculation yields that

Lemma 3.2. *We have*

$$\sigma(J) = -\frac{1}{2}J + \frac{9}{2}E, \quad \sigma(E) = -\frac{1}{6}J - \frac{1}{2}E.$$

Let

$$X^1 = J - \sqrt{27}iE,\ \ X^2 = J + \sqrt{27}iE. \tag{3.1}$$

Then it is easy to check that

$$\sigma(X^1) = \frac{-1+\sqrt{3}i}{2}X^1, \quad \sigma(X^2) = \frac{-1-\sqrt{3}i}{2}X^2. \tag{3.2}$$

It follows that $(V_{\mathbb{Z}\beta}^+)_4^{\langle\sigma\rangle} \subset L(1,0)$ and

$$(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle} = L(1,0)\bigoplus\sum_{n\geq 3} a_n L(1,n^2)$$

as a module for $L(1,0)$, where a_n is the multiplicity of $L(1,n^2)$ in $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$. By (3.2) we immediately have for any $n \in \mathbb{Z}$,

$$X_n^1 X^2 \in (V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}.$$

For convenience, we call highest weight vectors for the Virasoro algebra primary vectors. Note from [DG] that $V_{\mathbb{Z}\beta}^+$ contains two linearly independent primary vectors J and E of weight 4 and one linearly independent primary vector of weight 9. It is straightforward to compute that

$$J_3J = -72L(-4)\mathbf{1} + 336L(-2)^2\mathbf{1} - 60J,\ \ E_3E = -\frac{8}{3}L(-4)\mathbf{1} + \frac{112}{9}L(-2)^2\mathbf{1} + \frac{20}{9}J$$

(cf. [DJ3]). By Theorem 2.8 and Lemma 3.1, we have for $n \in \mathbb{Z}$

$$X_n^1X^2 \in L(1,0) \oplus L(1,9) \oplus L(1,16).$$

Note that $\sigma(E_{-2}J - J_{-2}E) = E_{-2}J - J_{-2}E$. Since $E_{-2}J - J_{-2}E \in M(1)\otimes e^{\beta} + M(1)\otimes e^{-\beta}$ we see immediately that $E_{-2}J - J_{-2}E$ is a primary vector of weight 9.

The following lemma follows from Theorem 3 in [DM1] and (3.2).

Lemma 3.3. *We have decomposition*

$$V_{\mathbb{Z}\beta}^+ = (V_{\mathbb{Z}\beta}^+)^0 \oplus (V_{\mathbb{Z}\beta}^+)^1 \oplus (V_{\mathbb{Z}\beta}^+)^2,$$

where $(V_{\mathbb{Z}\beta}^+)^0 = (V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$ *is a simple vertex operator algebra and* $(V_{\mathbb{Z}\beta}^+)^i$ *is the irreducible* $(V_{\mathbb{Z}\beta}^+)^0$*-module generated by* X^i *with lowest weight* 4, $i = 1, 2$.

Set

$$u^0 = -\frac{8}{3}L(-4)\mathbf{1} + \frac{112}{9}L(-2)^2\mathbf{1} \tag{3.3}$$

$$u^1 = -\frac{16}{9}L(-5)\mathbf{1} + \frac{112}{9}L(-3)L(-2)\mathbf{1} \tag{3.4}$$

$$u^2 = (-\frac{1856}{135}L(-6) - \frac{2384}{135}L(-4)L(-2) + \frac{1316}{135}L(-3)^2 + \frac{1088}{135}L(-2)^3)\mathbf{1} \tag{3.5}$$

$$u^3 = (-\frac{464}{45}L(-7) - \frac{928}{45}L(-5)L(-2) + \frac{40}{9}L(-4)L(-3) + \frac{544}{45}L(-3)L(-2)^2)\mathbf{1} \tag{3.6}$$

$$v^2 = (\frac{28}{75}L(-2) + \frac{23}{300}L(-1)^2)J, \tag{3.7}$$
$$v^3 = (\frac{14}{75}L(-3) + \frac{14}{75}L(-2)L(-1) - \frac{1}{300}L(-1)^3)J. \tag{3.8}$$

$$v^4 = (\frac{28}{75}L(-2) + \frac{23}{300}L(-1)^2)E, \tag{3.9}$$
$$v^5 = (\frac{14}{75}L(-3) + \frac{14}{75}L(-2)L(-1) - \frac{1}{300}L(-1)^3)E. \tag{3.10}$$

By Lemma 2.5 of [DJ3], we have the following lemma.

Lemma 3.4. *Let E and J be as before. Then*

$$E_3E = u^0 + \frac{20}{9}J,\ J_3J = 27u^0 - 60J,\ J_3E = 60E,$$
$$E_2E = u^1 + \frac{10}{9}L(-1)J,\ J_2J = 27u^1 - 30L(-1)J,\ J_2E = 30L(-1)E,$$
$$E_1E = u^2 + \frac{20}{9}v^2,\ J_1J = 27u^2 - 60v^2,\ J_1E = 60v^4,$$
$$E_0E = u^3 + \frac{20}{9}v^3,\ J_0J = 27u^3 - 60v^3,\ J_0E = 60v^5.$$

Using Lemma 3.4, one can check directly that

$$(J_{-2}E - E_{-2}J)_8J = -10800E,\ (J_{-2}E - E_{-2}J)_8E = 400J. \tag{3.11}$$

As a result, we have

$$(J_{-2}E - E_{-2}J)_8X^1 = -400\sqrt{27}iX^1,\ (J_{-2}E - E_{-2}J)_8X^2 = 400\sqrt{27}iX^2. \tag{3.12}$$

By (3.12), we immediately know that $J_{-2}E - E_{-2}J$ is a non-zero primary vector of weight 9. Recall from [DG] that $V_{\mathbb{Z}\beta}^+$ has one primary vector of weight 9 up to a constant. A direct calculation yields that

Lemma 3.5. *The vector*

$$\begin{aligned} u^{(9)} = & -\frac{1}{\sqrt{2}}(15h(-4)h(-1) + 10h(-3)h(-2) + 10h(-2)h(-1)^3) \otimes (e^{\beta} + e^{-\beta}) \\ & + (6h(-5) + 10h(-3)h(-1)^2 + \frac{15}{2}h(-2)^2h(-1) + h(-1)^5) \otimes (e^{\beta} - e^{-\beta}) \end{aligned}$$

is a non-zero primary vector of weight 9 *and* $u^{(9)} \in \mathbb{C}(J_{-2}E - E_{-2}J)$.

Note from [L1] that there is a non-degenerate symmetric invariant bilinear form $(\cdot,\cdot)$ on $V_{\mathbb{Z}\beta}^+$. The next lemma gives a relation between $u^{(9)}$ and $J_{-2}E - E_{-2}J$.

Lemma 3.6. *We have*

$$J_{-2}E - E_{-2}J = -2\sqrt{2}u^{(9)},$$
$$(u^{(9)}, u^{(9)}) = 5400.$$

Proof: By Lemma 3.5 and (3.11), we have $u_8^{(9)}E \in \mathbb{C}J$. Denote $F = e^{\beta} - e^{-\beta}$. Note that

$$(V_{\mathbb{Z}\beta}^+)_4 = \mathbb{C}h(-3)h(-1)\mathbf{1} \oplus \mathbb{C}h(-2)^2\mathbf{1} \oplus \mathbb{C}h(-1)^4\mathbf{1} \oplus \mathbb{C}E.$$

Let W_4 be the subspace of $(V_{\mathbb{Z}\beta}^+)_4$ linearly spanned by E, $h(-3)h(-1)\mathbf{1}$ and $h(-2)^2\mathbf{1}$. Then

$$h(-1)^4\mathbf{1} \equiv J \ (\text{mod } W_4).$$

Furthermore, we have

$$\begin{aligned}
(h(-4)h(-1)E)_8E \equiv& \sum_{i=0}^{\infty}(-1)^{i+1}\begin{pmatrix} -4 \\ i \end{pmatrix}(h(-1)E)_{4-i}h(i)E(\text{mod } W_4)\\
\equiv& -\sqrt{8}(h(-1)E)_4F(\text{mod } W_4)\\
\equiv& -\sqrt{8}\sum_{i=0}^{\infty}(-1)^{i}\begin{pmatrix} -1 \\ i \end{pmatrix}(h(-1-i)E_{4+i} + E_{3-i}h(i))F(\text{mod } W_4)\\
\equiv& -\sqrt{8}h(-1)E_4F - 8E_3E \ (\text{mod } W_4).
\end{aligned}$$

Similarly,

$$\begin{aligned}
(h(-3)h(-2)E)_8E \equiv& -8E_3E \ (\text{mod } W_4),\\
(h(-2)h(-1)^3E)_8E \equiv& -\sqrt{8}h(-1)^3E_6F - 24h(-1)^2E_5E\\
& - 24\sqrt{8}h(-1)E_4F - 64E_3E \ (\text{mod } W_4),\\
(h(-5)F)_8E \equiv& \sqrt{8}F_3F \ (\text{mod } W_4),\\
(h(-3)h(-1)^2F)_8E \equiv& \sqrt{8}h(-1)^2F_5F + 16h(-1)F_4E + 8\sqrt{8}F_3F \ (\text{mod } W_4).\\
(h(-2)^2h(-1)F)_8E \equiv& 8h(-1)F_4E + 8\sqrt{8}F_3F \ (\text{mod } W_4),\\
(h(-1)^5F)_8E \equiv& 5\sqrt{8}h(-1)^4F_7F + 80h(-1)^3F_6E + 80\sqrt{8}h(-1)^2F_5F\\
& + 320h(-1)F_4E + 64\sqrt{8}F_3F \ (\text{mod } W_4).
\end{aligned}$$

It is then easy to check that

$$u_8^{(9)}E = -100\sqrt{2}h(-1)^4\mathbf{1} \ (\text{mod } W_4).$$

This implies that

$$u_8^{(9)}E = -100\sqrt{2}J.$$

Then by (3.11),

$$J_{-2}E - E_{-2}J = -2\sqrt{2}u^{(9)}.$$

Note that

$$(J_{-2}E - E_{-2}J, J_{-2}E) = (J_8(J_{-2}E - E_{-2}J), E) = -((J_{-2}E - E_{-2}J)_8 J, E)$$

and

$$(J_{-2}E - E_{-2}J, E_{-2}J) = (E_8(J_{-2}E - E_{-2}J), J) = (-(J_{-2}E - E_{-2}J)_8 E, J).$$

Since

$$(E, E) = 2, \ (J, J) = 54,$$

(see [DJ2]) it follows from (3.11) that

$$(J_{-2}E - E_{-2}J, J_{-2}E - E_{-2}J) = 43200 \tag{3.13}$$

and

$$(u^{(9)}, u^{(9)}) = 5400.$$

The proof is complete. □

4 C_2-cofiniteness and rationality of $V_{L_2}^{A_4}$

The C_2-cofiniteness and rationality of $V_{L_2}^{A_4}$ is established in this section. The proof involves some very hard computations.

By Lemma 3.2, we have

$$J_{-9}J + 27E_{-9}E \in (V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}.$$

Then it is clear that

$$J_{-9}J + 27E_{-9}E = x^0 + X^{(16)} + 27(e^{2\beta} + e^{-2\beta}), \tag{4.1}$$

where $x^0 \in L(1,0)$, and $X^{(16)}$ is a non-zero primary element of weight 16 in $M(1)^+$. Denote

$$u^{(16)} = X^{(16)} + 27(e^{2\beta} + e^{-2\beta}). \tag{4.2}$$

Then $u^{(16)} \in (V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$ is a non-zero primary vector of weight 16.

Lemma 4.1. *We have the following:*

$$u_1^{(9)} u^{(9)} - 58800 u^{(16)} \in L(1,0).$$

Proof: Denote $E^2 = e^{2\beta} + e^{-2\beta}$. By Theorem 2.8 and the skew-symmetry, we may assume that

$$u_1^{(9)} u^{(9)} = v + c u^{(16)},$$

for some $v \in L(1,0)$ and $c \in \mathbb{C}$. To determine c we just need to consider $(u^{(9)}_1 u^{(9)}, E^2)$ by (4.2). Recall that

$$\begin{aligned} u^{(9)} = & -\frac{1}{\sqrt{2}}(15h(-4)h(-1) + 10h(-3)h(-2) + 10h(-2)h(-1)^3) \otimes E \\ & +(6h(-5) + 10h(-3)h(-1)^2 + \frac{15}{2}h(-2)^2h(-1) + h(-1)^5) \otimes F, \end{aligned}$$

where $F = e^{\beta} - e^{-\beta}$. To calculate $((h(-4)h(-1)\otimes E)_1(h(-4)h(-1)\otimes E), E^2)$, we only need to consider the coefficient of the monomial E^2 in $(h(-4)h(-1) \otimes E)_1(h(-4)h(-1) \otimes E)$. Then direct calculation yields that

$$((h(-4)h(-1) \otimes E)_1(h(-4)h(-1)\otimes)E), E^2) = (972E^2, E^2).$$

Calculations for other monomials are similar. For example,

$$((h(-3)h(-2) \otimes E)_1(h(-2)h(-1)^3\otimes)E), E^2) = (304E^2, E^2).$$

Then one can check that

$$(u^{(9)}_1 u^{(9)}, E^2) = (1587600E^2, E^2).$$

It follows that $c = 58800$. $\square$

Lemma 4.2. *The following hold: (1) $(V^+_{\mathbb{Z}\beta})^{\langle\sigma\rangle}$ is generated by $u^{(9)}$.*

(2) $(V^+_{\mathbb{Z}\beta})^{<\sigma>}$ is linearly spanned by

$$L(-m_s)\cdots L(-m_1)u^{(9)}_n u^{(9)},\ L(-m_s)\cdots L(-m_1)w^p_{-k_p}\cdots w^1_{-k_1}w,$$

where $w, w^1, \cdots, w^p \in \{u^{(9)}, u^{(16)}\}$, $k_p \geq \cdots \geq k_1 \geq 2$, $n \in \mathbb{Z}$, $m_s \geq \cdots \geq m_1 \geq 1$, $s, p \geq 0$.

Proof: By Lemma 3.6, ω can be generated by $u^{(9)}$. It follows from [DGR] that $(V^+_{\mathbb{Z}\beta})^{\langle\sigma\rangle}$ is generated by $u^{(9)}$ and $u^{(16)}$. Then (1) follows from Lemma 4.1.

By (3.2) in [A1] and (3.3) in [A3], we have

$$M(1, 2\sqrt{2}m) = \bigoplus_{p=0}^{\infty} L(1, (2m+p)^2), \tag{4.3}$$

$$V^+_{\mathbb{Z}\beta} = M(1)^+ \bigoplus(\bigoplus_{m=1}^{\infty} M(1, 2\sqrt{2}m)) = M(1)^+ \bigoplus(\bigoplus_{m=1}^{\infty}(\bigoplus_{p=0}^{\infty} L(1, (2m+p)^2)). \tag{4.4}$$

By (4.4) the subspace U^1 linearly spanned by primary elements of weight 16 in $V^+_{\mathbb{Z}\beta}$ is three dimensional. Obviously U^1 is invariant under σ. Note that $e^{2\beta} + e^{-2\beta} \in U^1$. Consider the $M(1)^+$-submodule W of $V^+_{\mathbb{Z}\beta}$ generated by $e^{2\beta} + e^{-2\beta}$. If $e^{2\beta} + e^{-2\beta} \in (V^+_{\mathbb{Z}\beta})^{<\sigma>}$, then by the fusion rule of $M(1)^+$ (also see [DN2]), $J \in W \cdot W = \langle u_n v | u, v \in W, n \in \mathbb{Z}\rangle$. So

$J \in (V_{\mathbb{Z}\beta}^+)^{<\sigma>}$, which contradicts with Lemma 3.2. This implies that σ has an eigenvector in U^1 with eigenvalue not equal to 1. Since $\sigma^3 = 1$ and U^1 is a vector space over $\mathbb{C}$, it follows that both $\frac{-1+\sqrt{3}i}{2}$ and $\frac{-1-\sqrt{3}i}{2}$ occur as eigenvalues of σ on U_1. Recall that $u^{(16)} \in (V_{\mathbb{Z}\beta}^+)^{<\sigma>}$ is a non-zero primary element of weight 16. So we immediately have

$$(V_{\mathbb{Z}\beta}^+)^{<\sigma>} = L(1,0) \bigoplus L(1,9) \bigoplus L(1,16) \bigoplus (\sum_{n\geq 5} a_n L(1,n^2)).$$

Let U^2 be the subspace of primary vectors of weight 25 in $V_{\mathbb{Z}\beta}^+$. By (4.4), $\dim U^2 = 2$. Consider the $M(1)^+$-submodule W of $V_{\mathbb{Z}\beta}^+$ generated by $e^{2\beta} + e^{-2\beta}$ again. By (4.3), there is a non-zero primary element $w = x \otimes (e^{2\beta} + e^{-2\beta}) + y \otimes (e^{2\beta} - e^{-2\beta})$ of weight 25 in W for some $x \in M(1)^+$ and $y \in M(1)^-$. Let $U^{(9)}$ and $U^{(16)}$ be the $L(1,0)$-submodules of $(V_{\mathbb{Z}\beta}^+)^{<\sigma>}$ generated by $u^{(9)}$ and $u^{(16)}$, respectively. Then by Part (1) and the skew-symmetry, any element of weight 25 in $(V_{\mathbb{Z}\beta}^+)^{<\sigma>}$ is a linear combination of elements in $L(1,0) \oplus U^{(9)} \oplus U^{(16)}$ and $U^{(9)} \cdot U^{(16)} =< u_n v | u \in U^{(9)}, v \in U^{(16)} >$. By Lemma 3.5 and (4.2), elements in $U^{(9)} \cdot U^{(16)}$ have the forms: $u \otimes (e^{\beta} + e^{-\beta}) + v \otimes (e^{\beta} - e^{-\beta})$, where $u \in M(1)^+$ and $v \in M(1)^-$. So we know that $w \notin (V_{\mathbb{Z}\beta}^+)^{<\sigma>}$. This proves that $\sigma|_{U^2}$ has eigenvalues not equal to 1. Since $\sigma^3 = 1$ and $\dim_{\mathbb{C}} U^2 = 2$, it follows that $\sigma|_{U^2}$ has two eigenvalues $\frac{-1+\sqrt{3}i}{2}$ and $\frac{-1-\sqrt{3}i}{2}$. So we immediately have

$$(V_{\mathbb{Z}\beta}^+)^{<\sigma>} = L(1,0) \bigoplus L(1,9) \bigoplus L(1,16) \bigoplus (\sum_{n\geq 6} a_n L(1,n^2)). \tag{4.5}$$

A proof similar to that of Lemma 4.3 in [DN2] gives (2) with the help of (1), (4.5) and Theorem 2.8. $\square$

Lemma 4.3. *We have*

$$\begin{aligned} u_{-3}^{(9)} u^{(9)} =& s^1 + \frac{162770}{99} L(-4)u^{(16)} + \frac{5204015}{1584} L(-3)L(-1)u^{(16)} + \frac{14760}{11} L(-2)^2 u^{(16)} \\ &+ \frac{1154225}{792} L(-2)L(-1)^2 u^{(16)} + \frac{354895}{3168} L(-1)^4 u^{(16)}, \end{aligned}$$

$$\begin{aligned} u_{-5}^{(9)} u^{(9)} =& s^2 - \frac{653871670}{63063 \cdot 27} L(-6)u^{(16)} + \frac{3303230375}{2018016 \cdot 27} L(-5)L(-1)u^{(16)} \\ &+ \frac{489993820}{63063 \cdot 27} L(-4)L(-2)u^{(16)} + \frac{69658220}{9009 \cdot 27} L(-3)^2 u^{(16)} \\ &+ \frac{346772585}{42042 \cdot 27} L(-4)L(-1)^2 u^{(16)} + \frac{3338006885}{168168 \cdot 27} L(-3)L(-2)L(-1)u^{(16)} \\ &+ \frac{19408720}{7007 \cdot 27} L(-2)^3 u^{(16)} + \frac{14067649205}{4036032 \cdot 27} L(-3)L(-1)^3 u^{(16)} \\ &+ \frac{1055175305}{252252 \cdot 27} L(-2)^2 L(-1)^2 u^{(16)} + \frac{1185150565}{2018016 \cdot 27} L(-2)L(-1)^4 u^{(16)} \\ &+ \frac{119070745}{8072064 \cdot 27} L(-1)^6 u^{(16)}, \end{aligned}$$

where $s^1, s^2 \in L(1,0)$.

Proof: By Theorem 2.8 and the skew-symmetry, we may assume that

$$u^{(9)}_{-3}u^{(9)} = s^1 + y^1,\ u^{(9)}_{-5}u^{(9)} = s^2 + y^2,$$

where $s^1, s^2 \in L(1,0)$, $y^1, y^2 \in U^{(16)} \cong L(1,16)$ which is an $L(1,0)$-submodule of $(V^+_{\mathbb{Z}\beta})^{\langle\sigma\rangle}$ generated by $u^{(16)}$. Then we may assume that

$$\begin{aligned} y^1 =& a_1L(-4)u^{(16)} + a_2L(-3)L(-1)u^{(16)} + a_3L(-2)^2u^{(16)} \\ &+ a_4L(-2)L(-1)^2u^{(16)} + a_5L(-1)^4u^{(16)} \\ =& \sum_{i=1}^{5} a_iw^i. \end{aligned}$$

To determine $a_i, 1 \le i \le 5$, we consider $(u^{(9)}_{-3}u^{(9)}, w^i)$, (w^i, w^j), $i, j = 1, 2, \cdots, 5$. Then by Lemma 4.1 and direct calculation, we have

$$\begin{bmatrix} 133 & 224 & 387 & 576 & 1920 \\ 224 & 3328 & 480 & 10560 & 49920 \\ 387 & 480 & 17673/2 & 13152 & 57600 \\ 576 & 10560 & 13152 & 162336 & 1267200 \\ 1920 & 49920 & 57600 & 1267200 & 30159360 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = 58800 \begin{bmatrix} 43 \\ 560 \\ 675 \\ 7344 \\ 93024 \end{bmatrix}.$$

We get that

$$a_1 = \frac{162770}{99},\ a_2 = \frac{5204015}{1584},$$
$$a_3 = \frac{14760}{11},\ a_4 = \frac{1154225}{792},\ a_5 = \frac{354895}{3168}.$$

The first formula follows. The proof for the second one is similar. We omit it. $\square$

Let v be any element in $V^+_{\mathbb{Z}\beta}$ of weight $m \le 22$. Then v is a linear combination of an element in $V^{(4)} \oplus V^{(16)}$ and elements in $M(1)^+$ having the form $h(-n_t)\cdots h(-n_1)\mathbf{1}$ such that $n_t \ge \cdots \ge n_1 \ge 1$ and $\sum_{i=1}^{t} n_i = m$, where $V^{(4)}$ and $V^{(16)}$ are $M(1)^+$-submodules of $V^+_{\mathbb{Z}\beta}$ generated by E and E^2 respectively. We denote by $c(v)$ the coefficient of the monomial $h(-1)^m\mathbf{1}$ in the linear combination. Then we have the following lemma.

Lemma 4.4.

$$c(u^{(9)}_{-3}u^{(9)}) = -\frac{447232}{19 \cdot 17 \cdot 11 \cdot 7^2 \cdot 5^2 \cdot 3},$$
$$c(u^{(9)}_{-5}u^{(9)}) = -\frac{328099328}{19 \cdot 17 \cdot 13 \cdot 11^2 \cdot 7^3 \cdot 5^2 \cdot 3^6}.$$

Proof: Let $k \in 2\mathbb{Z}+1$. We consider $c(u^{(9)}_{-k}u^{(9)})$. By a direct but long calculation, we have

$$\begin{aligned} c(u^{(9)}_{-k}u^{(9)}) = &-2700c(E_{-k-10}E) - 13500c(h(-1)^2E_{-k-8}E) - 18000\sqrt{2}c(h(-1)^3E_{-k-7}F) \\ &- 31500c(h(-1)^4E_{-k-6}E) - 15300\sqrt{2}c(h(-1)^5E_{-k-5}F) \\ &- 9060c(h(-1)^6E_{-k-4}E) - 1620\sqrt{2}c(h(-1)^7E_{-k-3}F) \\ &- 345c(h(-1)^8E_{-k-2}E) - 20\sqrt{2}c(h(-1)^9E_{-k-1}F) - c(h(-1)^{10}E_{-k}E) \end{aligned}$$

Note that for $m \in 2\mathbb{Z}+1$, $n \in 2\mathbb{Z}$, $m, n \leq 7$,

$$c(E_mE) = \frac{2\cdot(\sqrt{8})^{7-m}}{(7-m)!},\ c(E_nF) = -\frac{2\cdot(\sqrt{8})^{7-n}}{(7-n)!}.$$

Let $k = -3, k = -5$ respectively, we then get the lemma. □

As defined in [Z], a vertex operator algebra V is called C_2-cofinite, if $V/C_2(V)$ is finite-dimensional, where $C_2(V) = span_{\mathbb{C}}\{u_{-2}v | u, v \in V\}$. The following lemma comes from [Z].

Lemma 4.5. *(1) $L(-1)u \in C_2(V)$ for $u \in V$;*
(2) $u_{-k}v \in C_2(V)$, for $u, v \in V$ and $k \geq 2$;
(3) $u_{-1}v \in C_2(V)$, for $u \in V, v \in C_2(V)$.

We are now in a position to state the main result of this section.

Theorem 4.6. *$(V^+_{\mathbb{Z}\beta})^{\langle\sigma\rangle}$ is C_2-cofinite and rational.*

Proof: Let $s^1, s^2 \in L(1,0)$ be the same as in Lemma 4.3. Then s^1 and s^2 are linear combinations of linearly independent elements having the forms $L(-m_s)\cdots L(-m_1)\mathbf{1}$ and $L(-n_t)\cdots L(-n_1)\mathbf{1}$ respectively such that $m_s \geq \cdots \geq m_1 \geq 2$, $n_t \geq \cdots \geq n_1 \geq 2$ and $\sum_{i=1}^{s} m_i = 20$, $\sum_{i=1}^{t} n_i = 22$. Assume the coefficients of $L(-2)^{10}\mathbf{1}$ and $L(-2)^{11}\mathbf{1}$ in the two linear combinations are a_1 and a_2 respectively. Then by Lemma 4.5,

$$s^1 - a_1L(-2)^{10}\mathbf{1},\ s^2 - a_2L(-2)^{11}\mathbf{1} \in C_2((V^+_{\mathbb{Z}\beta})^{\langle\sigma\rangle}).$$

Further, by Lemma 4.5 and Lemma 4.3, we have

$$s^1 + \frac{14760}{11}L(-2)^2u^{(16)},\ s^2 + \frac{19408720}{7007\cdot 27}L(-2)^3u^{(16)} \in C_2((V^+_{\mathbb{Z}\beta})^{\langle\sigma\rangle}).$$

So

$$a_1L(-2)^{10}\mathbf{1} + \frac{14760}{11}L(-2)^2u^{(16)},\ a_2L(-2)^{11}\mathbf{1} + \frac{19408720}{7007\cdot 27}L(-2)^3u^{(16)} \in C_2((V^+_{\mathbb{Z}\beta})^{\langle\sigma\rangle}).$$

Thus by Lemma 4.5

$$a_1L(-2)^{11}\mathbf{1} + \frac{14760}{11}L(-2)^3u^{(16)},\ a_2L(-2)^{11}\mathbf{1} + \frac{19408720}{7007\cdot 27}L(-2)^3u^{(16)} \in C_2((V^+_{\mathbb{Z}\beta})^{\langle\sigma\rangle}). \tag{4.6}$$

On the other hand, note from the definition of $L(-2)\mathbf{1}$ that $c(L(-2)^k\mathbf{1}) = 2^k$. This implies that

$$c(u_{-3}^{(9)}u^{(9)}) = \frac{1}{2^{10}}a_1 + \frac{1}{4}\cdot\frac{14760}{11}c(X^{(16)}),$$
$$c(u_{-5}^{(9)}u^{(9)}) = \frac{1}{2^{11}}a_2 + \frac{1}{8}\cdot\frac{19408720}{7007\cdot 27}c(X^{(16)}).$$

So by Lemma 4.4,

$$\frac{1}{2^{10}}a_1 + \frac{1}{4}\frac{14760}{11}c(X^{(16)}) = -\frac{447232}{19\cdot 17\cdot 11\cdot 7^2\cdot 5^2\cdot 3}, \tag{4.7}$$

$$\frac{1}{2^{11}}a_2 + \frac{1}{8}\frac{19408720}{7007\cdot 27}c(X^{(16)}) = -\frac{328099328}{19\cdot 17\cdot 13\cdot 11^2\cdot 7^3\cdot 5^2\cdot 3^6}. \tag{4.8}$$

If

$$a_1/a_2 = \frac{14760}{11}/\frac{19408720}{7007\cdot 27},$$

then by (4.7) and (4.8), we have

$$\frac{-\frac{447232}{2\cdot 19\cdot 17\cdot 11\cdot 7^2\cdot 5^2\cdot 3}}{-\frac{328099328}{19\cdot 17\cdot 13\cdot 11^2\cdot 7^3\cdot 5^2\cdot 3^6}} = \frac{\frac{14760}{11}}{\frac{19408720}{7007\cdot 27}}.$$

But

$$\frac{-\frac{447232}{2\cdot 19\cdot 17\cdot 11\cdot 7^2\cdot 5^2\cdot 3}}{-\frac{328099328}{19\cdot 17\cdot 13\cdot 11^2\cdot 7^3\cdot 5^2\cdot 3^6}} = \frac{32688117}{2563276} \neq \frac{6346431}{485218} = \frac{\frac{14760}{11}}{\frac{19408720}{7007\cdot 27}}.$$

This means that

$$a_1/a_2 \neq \frac{14760}{11}/\frac{19408720}{7007\cdot 27}.$$

By (4.6), we have

$$L(-2)^{11}\mathbf{1}, L(-2)^3u^{(16)} \in C_2((V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}).$$

Then it follows from Lemma 4.2 that $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$ is C_2-cofinite. Since $V_{\mathbb{Z}\beta}^+$ is rational and $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$ is self-dual, it follows that $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$ satisfies the Hypothesis I in [M2]. Then by Corollary 7 in [M2], $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$ is rational. □

5 Classification and construction of irreducible modules of $(V_{\mathbb{Z}\beta}^+)^{<\sigma>}$

We will first construct all the irreducible σ^i-twisted modules of $V_{\mathbb{Z}\beta}^+$, $i = 1, 2$. We have the following lemma.

Lemma 5.1. *There are at most two inequivalent irreducible σ-twisted modules of $V_{\mathbb{Z}\beta}^+$.*

Proof: Let (W, Y) be an irreducible $V_{\mathbb{Z}\beta}^+$-module. Define a linear map

$$Y^\sigma : V_{\mathbb{Z}\beta}^+ \to (\mathrm{End}W)[[z, z^{-1}]]$$

by

$$Y^\sigma(u, z)w = Y(\sigma^{-1}(u), z)w$$

where $u \in V_{\mathbb{Z}\beta}^+$, $w \in W$. Recall from [DLM1] that (W, Y^σ) is still an irreducible module of $V_{\mathbb{Z}\beta}^+$, which we denote by W^σ. As in [DLM1], if $W \cong W^\sigma$, we say W is stable under σ. Recall from [DN2] that all the irreducible modules of $V_{\mathbb{Z}\beta}^+$ are

$$V_{\mathbb{Z}\beta}^\pm,\ V_{\mathbb{Z}\beta+\frac{r}{8}\beta}\ (1 \le r \le 3),\ V_{\mathbb{Z}\beta+\frac{\beta}{2}}^\pm,\ V_{\mathbb{Z}\beta}^{T_1,\pm},\ V_{\mathbb{Z}\beta}^{T_2,\pm}$$

with the following tables

	$V_{\mathbb{Z}\beta}^+$	$V_{\mathbb{Z}\beta}^-$	$V_{\mathbb{Z}\beta+\frac{1}{8}\beta}$	$V_{\mathbb{Z}\beta+\frac{1}{4}\beta}$	$V_{\mathbb{Z}\beta+\frac{3}{8}\beta}$	$V_{\mathbb{Z}\beta+\frac{\beta}{2}}^+$	$V_{\mathbb{Z}\beta+\frac{\beta}{2}}^-$
ω	0	1	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{9}{16}$	1	1
E	0	0	0	0	0	1	-1
J	0	-6	$-\frac{3}{64}$	0	$\frac{45}{64}$	3	3

	$V_{\mathbb{Z}\beta}^{T_1,+}$	$V_{\mathbb{Z}\beta}^{T_1,-}$	$V_{\mathbb{Z}\beta}^{T_2,+}$	$V_{\mathbb{Z}\beta}^{T_2,-}$
ω	$1/16$	$9/16$	$1/16$	$9/16$
E	$1/128$	$-15/128$	$-1/128$	$15/128$
J	$3/128$	$-45/128$	$3/128$	$-45/128$

It is easy to check that

$$V_{\mathbb{Z}\beta}^+ \cong (V_{\mathbb{Z}\beta}^+)^\sigma,\ (V_{\mathbb{Z}\beta+\frac{\beta}{4}})^\sigma \cong V_{\mathbb{Z}\beta+\frac{\beta}{4}},$$

$$(V_{\mathbb{Z}\beta}^-)^\sigma \cong V_{\mathbb{Z}\beta+\frac{\beta}{2}}^-,\ (V_{\mathbb{Z}\beta+\frac{\beta}{2}}^-)^\sigma \cong V_{\mathbb{Z}\beta+\frac{\beta}{2}}^+,\ (V_{\mathbb{Z}\beta+\frac{\beta}{2}}^+)^\sigma \cong V_{\mathbb{Z}\beta}^-,$$

$$(V_{\mathbb{Z}\beta+\frac{\beta}{8}})^\sigma \cong V_{\mathbb{Z}\beta}^{T_2,+},\ (V_{\mathbb{Z}\beta}^{T_2,+})^\sigma \cong V_{\mathbb{Z}\beta}^{T_1,+},\ (V_{\mathbb{Z}\beta}^{T_1,+})^\sigma \cong V_{\mathbb{Z}\beta+\frac{\beta}{8}},$$

$$(V_{\mathbb{Z}\beta+\frac{3\beta}{8}})^\sigma \cong V_{\mathbb{Z}\beta}^{T_2,-},\ (V_{\mathbb{Z}\beta}^{T_2,-})^\sigma \cong V_{\mathbb{Z}\beta}^{T_1,-},\ (V_{\mathbb{Z}\beta}^{T_1,-})^\sigma \cong V_{\mathbb{Z}\beta+\frac{3\beta}{8}}.$$

Then the lemma follows from [A2], [Y] and Theorem 10.2 in [DLM1]. $\square$

Next we will prove that there are exactly two inequivalent irreducible σ-twisted $V_{\mathbb{Z}\beta}^+$-modules. We first construct irreducible σ-twisted V_{L_2}-modules. Let $x^i, i = 1, 2, 3$ be defined as in Section 3. Set

$$h' = \frac{1}{3\sqrt{6}}(x^1 + x^2 - x^3),$$

$$y^1 = \frac{1}{\sqrt{3}}(x^1 + \frac{-1+\sqrt{3}i}{2}x^2 + \frac{1+\sqrt{3}i}{2}x^3),$$

$$y^2 = \frac{1}{\sqrt{3}}(x^1 + \frac{-1-\sqrt{3}i}{2}x^2 + \frac{1-\sqrt{3}i}{2}x^3).$$

Then

$$L(n)h' = \delta_{n,0}h',\ h'(n)h' = \frac{1}{18}\delta_{n,1}\mathbf{1},\ n \in \mathbb{Z},$$

$$h'(0)y^1 = \frac{1}{3}y^1,\ h'(0)y^2 = -\frac{1}{3}y^2,\ y^1(0)y^2 = 6h'.$$

It follows that $h'(0)$ acts semisimply on V_{L_2} with rational eigenvalues. So $e^{2\pi i h'(0)}$ is an automorphism of V_{L_2} (see [L2], [DG], etc.). Since

$$e^{2\pi i h'(0)}h' = h',\ e^{2\pi i h'(0)}y^1 = \frac{-1+\sqrt{3}i}{2}y^1,\ e^{2\pi i h'(0)}y^2 = \frac{-1-\sqrt{3}i}{2}y^2,$$

it is easy to see that

$$e^{2\pi i h'(0)} = \sigma.$$

Let

$$\Delta(h', z) = z^{h'(0)} \exp(\sum_{k=1}^{\infty} \frac{h'(k)}{-k}(-z)^{-k}),$$

and

$$W^1 = V_{L_2},\ W^2 = V_{L_2+\frac{\alpha}{2}}.$$

Then W^1 and W^2 are all the irreducible V_{L_2}-modules and

$$W^1(0) = \mathbb{C}\mathbf{1},\ W^2(0) = \mathbb{C}e^{\frac{\alpha}{2}} \bigoplus \mathbb{C}e^{-\frac{\alpha}{2}}.$$

Let

$$w^1 = e^{\frac{\alpha}{2}} + \frac{(\sqrt{3}-1)(1+i)}{2}e^{-\frac{\alpha}{2}},$$

$$w^2 = \frac{1}{\sqrt{2}}[(\sqrt{3}-1)e^{\frac{\alpha}{2}} - (1+i)e^{-\frac{\alpha}{2}}].$$

Then $W^2 = \mathbb{C}w^1 \bigoplus \mathbb{C}w^2$ and

$$h'(0)w^1 = \frac{1}{6}w^1,\ h'(0)w^2 = -\frac{1}{6}w^2,$$

$$y^1(0)w^1 = 0,\ y^1(0)w^2 = w^1, y^2(0)w^1 = w^2.$$

From [L2], we have the following lemma.

Lemma 5.2. $(W^{i,T_1}, Y_\sigma(\cdot, z)) = (W^i, Y(\Delta(h', z)\cdot, z))$ *are irreducible* σ*-twisted modules of* $V_{L_2}, i = 1, 2$.

Direct calculation yields that

$$\Delta(h', z)L(-2)\mathbf{1} = L(-2)\mathbf{1} + z^{-1}h'(-1)\mathbf{1} + \frac{1}{36}z^{-2}\mathbf{1}, \tag{5.1}$$

$$Y_\sigma(h', z) = Y(h' + \frac{1}{18}z^{-1}, z), \tag{5.2}$$

$$Y_\sigma(y^1, z) = z^{\frac{1}{3}}Y(y^1, z), \tag{5.3}$$

$$Y_\sigma(y^2, z) = z^{-\frac{1}{3}}Y(y^2, z). \tag{5.4}$$

To distinguish the components of $Y(u, z)$ from those of $Y_\sigma(u, z)$ we consider the following expansions

$$Y_\sigma(u, z) = \sum_{n\in\mathbb{Z}+\frac{r}{3}} u_n z^{-n-1},\ Y(u, z) = \sum_{n\in\mathbb{Z}} u(n)z^{-n-1},$$

where $u \in V_{L_2}$ such that $\sigma(u) = e^{-\frac{2r\pi i}{3}}u$. By (5.2)-(5.4) and direct calculation, we have the following lemma.

Lemma 5.3. *Write $W^{i,T_1} = \oplus_{n\in\frac{1}{3}\mathbb{Z}_+} W^{i,T_1}(n)$ as admissible σ-twisted module. Then*

$$W^{1,T_1}(0) = \mathbb{C}\mathbf{1},\ W^{1,T_1}(\frac{1}{3}) = \mathbb{C}y^1_{-\frac{1}{3}}\mathbf{1} = 0,$$

$$W^{1,T_1}(\frac{2}{3}) = \mathbb{C}y^2_{-\frac{2}{3}}\mathbf{1} = \mathbb{C}y^2,\ W^{1,T_1}(\frac{4}{3}) = \mathbb{C}y^1_{-\frac{4}{3}}\mathbf{1} = \mathbb{C}y^1,$$

$$W^{2,T_1}(0) = \mathbb{C}w^2,\ W^{2,T_1}(\frac{1}{3}) = \mathbb{C}y^1_{-\frac{1}{3}}w^2 = \mathbb{C}w^1,$$

$$W^{2,T_1}(\frac{2}{3}) = \mathbb{C}y^2_{-\frac{2}{3}}w^2 = 0,\ W^{2,T_1}(\frac{5}{3}) = \mathbb{C}y^2_{-\frac{5}{3}}w^2 = \mathbb{C}y^2(-2)w^2,$$

$$L(0)|_{W^{1,T_1}(0)} = \frac{1}{36}id,\ L(0)|_{W^{2,T_1}(0)} = \frac{1}{9}id.$$

We have the following result.

Theorem 5.4. *W^{1,T_1} and W^{2,T_1} are the only two irreducible σ-twisted modules of $V_{\mathbb{Z}\beta}^+$.*

Proof: By Lemma 5.3, W^{1,T_1} and W^{2,T_1} are inequivalent σ-twisted modules of $V_{\mathbb{Z}\beta}^+$. Note that W^{1,T_1} and W^{2,T_1} have irreducible quotients which are σ-twisted modules of $V_{\mathbb{Z}\beta}^+$ with lowest weights $\frac{1}{36}$ and $\frac{1}{9}$, respectively. If W^{i,T_1} is not irreducible for some i, then the lowest weight λ of the maximal proper submodule is different from $\frac{1}{36}$ and $\frac{1}{9}$. By [Y], $V_{\mathbb{Z}\beta}^+$ is C_2-cofnite. It follows from [DLM1] that $V_{\mathbb{Z}\beta}^+$ has an irreducible σ-twisted module with lowest weight λ. This means that there are at least three inequivalent irreducible σ-twisted modules of $V_{\mathbb{Z}\beta}^+$, which contradicts Lemma 5.1. So W^{1,T_1} and W^{2,T_1} are irreducible inequivalent σ-twisted $V_{\mathbb{Z}\beta}^+$-modules. Then the theorem follows from Lemma 5.1. $\square$

Note that

$$\sigma^2 = \sigma^{-1} = e^{2\pi i(-h'(0))},$$

and

$$e^{2\pi i(-h'(0))}(-h') = -h',\ e^{2\pi i(-h'(0))}y^1 = \frac{-1-\sqrt{3}i}{2}y^1,\ e^{2\pi i(-h'(0))}y^2 = \frac{-1+\sqrt{3}i}{2}y^2.$$

So we similarly have

Lemma 5.5. $(W^{i,T_2}, Y_{\sigma^{-1}}(\cdot,z)) = (W^i, Y(\Delta(-h',z)\cdot,z))$ *are irreducible σ^{-1}-twisted modules of V_{L_2}, $i = 1,2$.*

It is easy to see that

$$\Delta(-h',z)L(-2)\mathbf{1} = L(-2)\mathbf{1} - z^{-1}h'(-1)\mathbf{1} + \frac{1}{36}z^{-2}\mathbf{1}, \tag{5.5}$$

$$Y_{\sigma^{-1}}(-h',z) = Y(-h' + \frac{1}{18}z^{-1}, z), \tag{5.6}$$

$$Y_{\sigma^{-1}}(y^1,z) = z^{-\frac{1}{3}}Y(y^1,z), \tag{5.7}$$

$$Y_{\sigma^{-1}}(y^2,z) = z^{\frac{1}{3}}Y(y^2,z). \tag{5.8}$$

By (5.5)-(5.8), we have

$$W^{1,T_2}(0) = \mathbb{C}\mathbf{1},\ W^{1,T_2}(\frac{1}{3}) = \mathbb{C}y^2_{-\frac{1}{3}}\mathbf{1} = 0,$$

$$W^{1,T_2}(\frac{2}{3}) = \mathbb{C}y^1_{-\frac{2}{3}}\mathbf{1} = \mathbb{C}y^1,\ W^{1,T_2}(\frac{4}{3}) = \mathbb{C}y^2_{-\frac{4}{3}}\mathbf{1} = \mathbb{C}y^2,$$

$$W^{2,T_2}(0) = \mathbb{C}w^1,\ W^{2,T_2}(\frac{1}{3}) = \mathbb{C}y^2_{-\frac{1}{3}}w^1 = \mathbb{C}w^2,$$

$$W^{2,T_2}(\frac{2}{3}) = \mathbb{C}y^1_{-\frac{2}{3}}w^1 = 0,\ W^{2,T_2}(\frac{5}{3}) = \mathbb{C}y^1_{-\frac{5}{3}}w^1 = \mathbb{C}y^1(-2)w^1,$$

$$L(0)|_{W^{1,T_2}(0)} = \frac{1}{36}id,\ L(0)|_{W^{2,T_2}(0)} = \frac{1}{9}id.$$

Similar to Theorem 5.4, we have

Theorem 5.6. *W^{1,T_2} and W^{2,T_2} are the only two irreducible σ^2-twisted modules of $V_{\mathbb{Z}\beta}^+$.*

We finally classify all the irreducible modules of $V_{L_2}^{A_4}$. Recall that $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle} = V_{L_2}^{A_4}$. We prove, in particular, that any irreducible $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$-module is contained in some irreducible σ^i-twisted $V_{\mathbb{Z}\beta}^+$-module, $i = 0,1,2$.

Let X^1 and X^2 be defined as in (3.1). By Lemma 3.3, X^i generates an irreducible $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$-module with lowest weight 4, denoted by $(V_{\mathbb{Z}\beta}^+)^i$, $i = 1,2$.

Note that W^{i,T_1}, W^{i,T_2}, $i=1,2$ can also be regarded as $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$-modules. Set

$$w^{1,T_1,1}=\mathbf{1}\in W^{1,T_1}(0),\ w^{1,T_1,2}=y^2\in W^{1,T_1}(\frac{2}{3}),\ w^{1,T_1,3}=y^1\in W^{1,T_1}(\frac{4}{3}),$$

$$w^{2,T_1,1}=w^2\in W^{2,T_1}(0),\ w^{2,T_1,2}=w^1\in W^{2,T_1}(\frac{1}{3}),\ w^{2,T_1,3}=y^2(-2)w^2\in W^{2,T_1}(\frac{5}{3}),$$

$$w^{1,T_2,1}=\mathbf{1}\in W^{1,T_2}(0),\ w^{1,T_2,2}=y^1\in W^{1,T_2}(\frac{2}{3}),\ w^{1,T_2,3}=y^2\in W^{1,T_1}(\frac{4}{3}),$$

$$w^{2,T_2,1}=w^1\in W^{2,T_2}(0),\ w^{2,T_2,2}=w^2\in W^{2,T_2}(\frac{1}{3}),\ w^{2,T_2,3}=y^1(-2)w^1\in W^{2,T_2}(\frac{5}{3}).$$

Then we have the following lemma.

Lemma 5.7. *Let $W^{i,T_j,k}$ be the $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$-module generated by $w^{i,T_j,k}$, where $i,j=1,2$, $k=1,2,3$. Then $W^{i,T_j,k}$, $i,j=1,2,k=1,2,3$ are irreducible $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$-modules such that*

$$L(0)w^{1,T_1,1}=\frac{1}{36}w^{1,T_1,1},\ L(0)w^{1,T_2,1}=\frac{1}{36}w^{1,T_2,1},$$

$$L(0)w^{1,T_1,2}=\frac{25}{36}w^{1,T_1,2},\ L(0)w^{1,T_2,2}=\frac{25}{36}w^{1,T_2,2},$$

$$L(0)w^{1,T_1,3}=\frac{49}{36}w^{1,T_1,3},\ L(0)w^{1,T_2,3}=\frac{49}{36}w^{1,T_2,3},$$

$$L(0)w^{2,T_1,1}=\frac{1}{9}w^{2,T_1,1},\ L(0)w^{2,T_2,1}=\frac{1}{9}w^{2,T_2,1},$$

$$L(0)w^{2,T_1,2}=\frac{4}{9}w^{2,T_1,2},\ L(0)w^{2,T_2,2}=\frac{4}{9}w^{2,T_2,2},$$

$$L(0)w^{2,T_1,3}=\frac{16}{9}w^{2,T_1,3},\ L(0)w^{2,T_2,3}=\frac{16}{9}w^{2,T_2,3}.$$

Proof: The lemma follows from a general result: Let U be a vertex operator algebra with an automorphism g of order T. Let $M=\sum_{n\in\frac{1}{T}\mathbb{Z}_+}M(n)$ be an irreducible g-twisted admissible U-module. Then $M^i=\oplus_{n\in\frac{i}{T}+\mathbb{Z}}M(n)$ is an irreducible V^g-module for $i=0,...,T-1$ (cf. [DM1]). $\square$

We have the following lemma from [DM1]

Lemma 5.8. *As an $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$-module,*

$$V_{\mathbb{Z}+\frac{1}{4}\beta}=(V_{\mathbb{Z}+\frac{1}{4}\beta})^0\oplus(V_{\mathbb{Z}+\frac{1}{4}\beta})^1\oplus(V_{\mathbb{Z}+\frac{1}{4}\beta})^2$$

such that $(V_{\mathbb{Z}+\frac{1}{4}\beta})^0$, $(V_{\mathbb{Z}+\frac{1}{4}\beta})^1$ and $(V_{\mathbb{Z}+\frac{1}{4}\beta})^2$ are irreducible $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$-modules generated by $e^{\beta/4}+e^{-\beta/4}$, $h(-2)\otimes(e^{\beta/4}-e^{-\beta/4})-\sqrt{2}h(-1)^2\otimes(e^{\beta/4}+e^{-\beta/4})+a(e^{3\beta/4}+e^{-3\beta/4})$ and $h(-2)\otimes(e^{\beta/4}-e^{-\beta/4})-\sqrt{2}h(-1)^2\otimes(e^{\beta/4}+e^{-\beta/4})-a(e^{3\beta/4}+e^{-3\beta/4})$ for some $0\neq a\in\mathbb{C}$ with weights $\frac{1}{4}$, $\frac{9}{4}$ and $\frac{9}{4}$ respectively.

We are now in a position to state the main result of this section. Recall that $(V_{\mathbb{Z}\beta}^+)^0 = (V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$.

Theorem 5.9. *There are exactly 21 irreducible modules of $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$. We give them by the following tables 1-4.*

	$(V_{\mathbb{Z}\beta}^+)^0$	$(V_{\mathbb{Z}\beta}^+)^1$	$(V_{\mathbb{Z}\beta}^+)^2$	$V_{\mathbb{Z}\beta}^-$	$V_{\mathbb{Z}\beta+\frac{1}{8}\beta}$	$V_{\mathbb{Z}\beta+\frac{3}{8}\beta}$
ω	0	4	4	1	$\frac{1}{16}$	$\frac{9}{16}$

	$W^{1,T_1,1}$	$W^{1,T_1,2}$	$W^{1,T_1,3}$	$W^{2,T_1,1}$	$W^{2,T_1,2}$	$W^{2,T_1,3}$
ω	$\frac{1}{36}$	$\frac{25}{36}$	$\frac{49}{36}$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{16}{9}$

	$W^{1,T_2,1}$	$W^{1,T_2,2}$	$W^{1,T_2,3}$	$W^{2,T_2,1}$	$W^{2,T_2,2}$	$W^{2,T_2,3}$
ω	$\frac{1}{36}$	$\frac{25}{36}$	$\frac{49}{36}$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{16}{9}$

	$(V_{\mathbb{Z}\beta+\frac{1}{4}\beta})^0$	$(V_{\mathbb{Z}\beta+\frac{1}{4}\beta})^1$	$(V_{\mathbb{Z}\beta+\frac{1}{4}\beta})^2$
ω	$\frac{1}{4}$	$\frac{9}{4}$	$\frac{9}{4}$

Proof: It follows from the proof of Lemma 5.1 and Theorem 6.1 in [DM1] that $V_{\mathbb{Z}\beta}^-$, $V_{\mathbb{Z}\beta+\frac{1}{8}\beta}$ and $V_{\mathbb{Z}\beta+\frac{3}{8}\beta}$ are irreducible $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$-modules and as $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$-modules,

$$V_{\mathbb{Z}\beta}^- \cong V_{\mathbb{Z}\beta+\frac{\beta}{2}}^- \cong V_{\mathbb{Z}\beta+\frac{\beta}{2}}^+,\ V_{\mathbb{Z}\beta+\frac{\beta}{8}} \cong V_{\mathbb{Z}\beta}^{T_2,+} \cong V_{\mathbb{Z}\beta}^{T_1,+},\ V_{\mathbb{Z}\beta+\frac{3\beta}{8}} \cong V_{\mathbb{Z}\beta}^{T_2,-} \cong V_{\mathbb{Z}\beta}^{T_1,-}.$$

Then the theorem follows from Lemma 3.3, Lemma 5.8, Theorems 5.4 and 5.6, Theorem 4.6 and Theorem A in [M1]. □

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